Markov numbers

The best introduction and discussion of the subject is in [4] Cassels, An Introduction to Diophantine Approximation, Chapter 2.

Markov numbers are defined to be the integers occurring as solutions of the Diophantine equation:

$$x^2 + y^2 + z^2 - 3xyz = 0 \qquad (1)$$

When talking about a triplet satisfying the equation (1), say $(x_0,\, y_0,\, z_0),$ we will always assume that

$$x_0 \leq y_0 \leq z_0 \qquad (2)$$

Theorem 1 : The only triplets satisfying (1), in which an equality occurs in (2), are $(1,\, 1,\, 1)$ and $(1,\, 1,\, 2)$

Proof.

(α) Assume $x_0 = y_0$. This leads to the equation in z:

$$2x^2 + z^2 - 3x^2 z = 0$$

Solving for z, we get

$$z = \frac{1}{2}\left(3x^2 \pm \sqrt{9x^4 - 8x^2}\right) = \frac{1}{2}\left(3x^2 \pm x\sqrt{9x^2 - 8}\right)$$

This requires that $9x^2 - 8$ be a perfect square, say t^2. Hence

$$9x^2 - 8 = t^2 \text{ or } 9x^2 - t^2 = 8$$

or

$$(3x - t)(3x + t) = 8 = 1 \cdot 8 = 2 \cdot 4$$

This leads to two possibilities:

$$(3x - t = 1 \text{ and } 3x + t = 8) \Rightarrow 6x = 9 \text{ i.e. no integer solution}$$

and

$$(3x - t = 2 \text{ and } 3x + 2 = 4) \Rightarrow x = 1$$

To solve for z, we plug $x = y = 1$. Then (1) becomes $2 + z^2 - 3z = 0,$ giving $z = 1$ or $z = 2$.

(β) Assume $y_0 = z_0$. This leads to the equation z

$$2y^2 + z^2 - 3y^2 z = 0$$

which is identical with case (α), except that y replaces x. So the only solutions in this case are also $(1,\ 1,\ 1)$ and (1, 1, 2) QED.

Assume $(x,\ y,\ z)$ is a solution of (1), with $x \leq y \leq z$. If we keep two of the variables fixed, this produces a quadratic equation in the third variable, with two solutions. Since the equation is monic with integer coefficients, the other solution is also an integer, call it x', y', or z', whichever is the case.

If

$$t^2 + bt + c = 0$$

is a quadratic equation in t, with roots r_1, and r_2 then

$$\begin{aligned} r_{1,\,2} &= \frac{1}{2}\Big(-b \pm \sqrt{b^2 - 4c}\Big) \\ r_1 + r_2 &= -b \\ r_1 \cdot r_2 &= c \end{aligned}$$

Thus, we have the following formulas:

$$x' = 3yz - x = \frac{y^2 + z^2}{x} = \frac{1}{2}\Big(3yz \pm \sqrt{9y^2z^2 - 4(y^2 + z^2)}\Big) \tag{3}$$

$$y' = 3xz - y = \frac{x^2 + z^2}{y} = \frac{1}{2}\Big(3xz \pm \sqrt{9x^2z^2 - 4(x^2 + z^2)}\Big) \tag{4}$$

$$z' = 3xy - z = \frac{x^2 + y^2}{z} = \frac{1}{2}\Big(3xy \pm \sqrt{9x^2y^2 - 4(x^2 + y^2)}\Big) \tag{5}$$

For example, the triplet of solutions (1, 2, 5) leads to three new solutions:

$(2,\ 5,\ 29)$ where $x' = 29$
$(1,\ 5,\ 13)$ where $y' = 13$
$(1,\ 1,\ 2)$ where $z' = 1$

We can actually figure out the signs in the formulas (3), (4), and (5):

Theorem 2. Assume that $x < y < z$. In the formulas (3) and (4) the sign is $+$, and in the formula (5) the sign is $-$.

Proof: This will follow from the following lemma:

Lemma 1. Suppose $1 \le t < z$ and $1 \le x < y$ are integers. Then

(α) $\frac{1}{2}\left(3tz + \sqrt{9t^2z^2 - 4(t^2+z^2)}\right) \ge z$

and

(β) $\frac{1}{2}\left(3xy - \sqrt{9x^2y^2 - 4(x^2+y^2)}\right) < y$

Proof: Part (α) is clear:

$$\frac{1}{2}\left(3tz + \sqrt{9t^2z^2 - 4(t^2+z^2)}\right) > \frac{3tz}{2} > z$$

As to part (β), let $\psi(x,\, y) = \frac{1}{2}\left(3xy - \sqrt{9x^2y^2 - 4(x^2+y^2)}\right)$.

$$\psi(x,\, y) = \frac{1}{2}3xy\left(1 - \sqrt{1 - \frac{4}{9}\left(\frac{1}{x^2} + \frac{1}{y^2}\right)}\right)$$

or, after a bit of algebra:

$$\psi(x,\, y) = \frac{1}{1 + \sqrt{1 - \frac{4}{9}\left(\frac{1}{x^2} + \frac{1}{y^2}\right)}} \cdot \frac{1}{2} \cdot 3xy \cdot \frac{4}{9}\left(\frac{1}{x^2} + \frac{1}{y^2}\right)$$

Now, the first fraction, the one with the square root in the denominator, assumes maximum when $x = 1$ and $y = 2$, because of the hypothesis on x and y. Thus, the fraction in question at most $\frac{3}{5}$. Hence

$$\psi(x,\, y) \le \frac{3}{5} \cdot \frac{1}{2} \cdot 3 \cdot \frac{4}{9} \cdot \left(\frac{y}{x} + \frac{x}{y}\right) \le \frac{2}{5}(y+1) < y. \text{ QED.}$$

Proof of Theorem 2: If "$-$" holds in (3) or (4), then

$$x \text{ or } y = \frac{1}{2}\left(3tz + \sqrt{9t^2z^2 - 4(t^2+z^2)}\right)$$

where $t = y$ or x, which would imply that $x > z$ or $y > z$, contrary to hypothesis. If "$+$" holds in (5) then

$$z = \frac{1}{2}\left(3xy - \sqrt{9x^2y^2 - 4(x^2+y^2)}\right) < y$$

again contradicting the hypothesis.

Definition 1. Let $(x,\, y,\, z)$ be a solution of (1) satisfying (2). Define

$$X(x,\, y,\, z) = (y,\, z,\, x')$$
$$Y(x,\, y,\, z) = (x,\, z,\, y')$$
$$Z(x,\, y,\, z) = (x,\, z',\, y) \text{ or } (z',\, x,\, y)$$

where x', y', and z' are given by the formulas (3), (4), and (5).

Theorem 3. If $x < y < z$ then

$$z < y' < x' \text{ and } z' < y$$

Proof: The facts that $z < y'$ and $z' < y$ are proved in Lemma 2. To show that $y' < x'$ we have by (3) and (4):

$$x' - y' = (3yz - x) - (3xz - y) = (3z + 1)(y - x) > 0. \text{ QED.}$$

Definition 2. If $(x,\, y,\, z)$ is a solution of (1) satisfying $x \leq y \leq z$, define

$$\|(x,\, y,\, z)\| = z$$

i.e., the largest of the numbers x, y, and z.

Theorem 4. (Markov [9]) Every solution of (1) can be obtained by starting with the triple (1, 1, 1) and repeatedly obtaining new solutions by one of the transformations X and/or Y.

Proof. One easily checks that all the solutions $(x,\, y,\, z)$ with $z < 10$, say, can be so obtained. Given a solution $(x,\, y,\, z)$, with $z \geq 10$, we have

$$Z(x,\, y,\, z) = (a,\, b,\, c) \text{ where}$$
$$(a,\, b,\, c) = (x,\, z',\, y) \text{ or } (z',\, x,\, y)$$

Now, $c = y < z$, and $(x,\, y,\, z) = X(a,\, b,\, c)$ or $(x,\, y,\, z) = Y(a,\, b,\, c)$. Continuing in this manner we get to a triple $(a,\, b,\, c)$ with $c < 10$, and the result follows. QED.

There is a conjecture due to Frobenius which is almost a century old, see [6]:

Given a Markov number z, the sequence of transformations of X's and Y's leading from $(1,\, 1,\, 1)$ to $(x,\, y,\, z)$ is unique.

Another way of restating this conjecture is:

If $(x_1,\, y_1,\, z)$ and $(x_2,\, y_2,\, z)$ are two solutions of (1), with $x_i \leq y_i \leq z$, then $x_1 = x_2$ and $y_1 = y_2$.

It is known that this is true when z is a power of a prime. See [1], [2], and [3] for the discussion and the proofs.

Markov numbers arise in the theory of approximations as follows. (See Cassels [4] for a complete discussion.) Suppose θ is a positive irrational numbers whose continued fraction expansion is:

$$\theta = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots \cfrac{1}{a_n + \ddots}}}} \tag{6}$$

which we will denote as

$$\theta = [a_0; a_1, a_2, \ldots, a_n, \ldots] \tag{7}$$

Assume that the "continued fraction digits" a_i of θ are bounded. Then there is a number $M(\theta)$ with the following properties:

(α) For ever$y$ $\epsilon > 0$ there are infinitely many pairs of integers p and q such that:

$$\left|\theta - \frac{p}{q}\right| < \frac{1+\epsilon}{M(\theta)q^2}$$

(β) For every $\epsilon > 0$ there are only finitely many pairs of integers p and q such that

$$\left|\theta - \frac{p}{q}\right| < \frac{1-\epsilon}{M(\theta)q^2}$$

In fact

$$M(\theta) = \limsup_{n \to \infty} [a_{n+1}; a_{n+2}, \ldots] = \limsup_{n \to \infty} a_n + \cfrac{1}{a_{n+1} + \cfrac{1}{a_{n+2} + \cfrac{1}{\ddots}}}$$

Moreover, let

$$\frac{p_n}{q_n} = [a_0;\, a_1,\, a_2,\, \ldots,\, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots \cfrac{1}{a_n}}} \tag{8}$$

be the continued fraction of θ truncated at a_n. The integers p and q in (α) are then actually given by (8).

The totality of all the numbers $M(\theta)$, where θ's is taken to be all irrationals with bounded continued fraction digits, is called **Markov spectrum**. Denote it by M. Its structure is topologically quite complicated, see Cusick and Flaheve [5] for a comprehensive discussion. The set M has many isolated points, in fact if

$$u_1, u_2, u_3,\, u_4,\, \ldots\, =\, 1,\, 2,\, 5,\, 13,\, 29,\, \ldots$$

is the sequence of all Markov numbers arranged in increasing order, then

$$\mu_n = \sqrt{9 - \frac{4}{u_n}}$$

is an isolated point of Markov spectrum. The numbers $\mu_n \uparrow 3$ and all the other points μ of the spectrum M satisfy $\mu \geq 3$. The structure of the set M above 3 is very complicated, in places it resembles the Cantor's set. There is however a point x, such that M contains all the numbers $\geq x$. See [5] and [7].

References:

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